P.G. Semester-II Examination, 2023

MATHEMATICS

Course ID : 22151 **Course Code : MATH201C**

Course Title : Complex Analysis

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

Answer any **five** questions from the following:

8×5=40

1. a) Let the stereographic projection of the point $P(\alpha, \beta, \gamma)$ be the point (3, 4) on the complex plane. Find the point *P*.

   b) Determine all possible transformations associated with a bilinear transformation.

   c) Suppose that a function *f* is analytic in the domain *D* which contains a segment of the *x*-axis and whose lower half is the reflection of the upper half with respect to the axis. Prove that $\overline{f(z)} = f(\overline{z})$ for each point *z* in the domain if *f* is real for each point *x* on the segment. 2+2+4

[Turn Over]

2. a) Using maximum modulus theorem, prove that $|f(z)| \leq 6$ on $|z| \leq 2$ where $f(z) = z - 3i$.

   b) Suppose *f* is analytic in a domain *D*, and $\{z_n\}$ is a sequence of distinct points converging to $z_0 \in D$. If $f(z_n) = 0$ for each $n \in \mathbb{N}$, prove that $f(z) \equiv 0$ everywhere in *D*.

   c) Find the bilinear transformation which transforms the points *z* = 2, 1, 0 into *w* = 1, 0, *i* respectively.

   3+2+3

3. a) Let *f* be an analytic function over the domain *D* of the *z*-plane. Prove that the magnitude and direction of rotation of the angle both are preserved under the transformation $w = f(z)$.

   b) Let $w = \dfrac{az+b}{cz+d}$ be a mobius transformation which transform the circle $|z| = 1$ into $|w| = 1$. The points 0, ∞ of *w*-plane correspond to the points α, $\dfrac{1}{\overline{\alpha}}$ of the *z*-plane. Show that $w = e^{i\lambda} \dfrac{z-\alpha}{z\overline{\alpha}-1}$.

   c) What is meant by ε-nbd of ∞? 4+3+1

4. a) Use the definition of contour integration to evaluate $\int_C \overline{z}\, dz$ where C is the positively oriented square with the end points 1, $1 + 2i$, $-1 + 2i$, -1.

b) Let f be a complex continuous function on a domain D such that the integrals of f around closed contours lying entirely in D are zero. Show that f has an antiderivative in D.

c) Let C denote the boundary of a triangle with vertices 0, $3i$, -4. Find an upper bound of $\left|\int_C \left(e^z - \overline{z}\right) dz\right|$. 3+3+2

5. a) If f is analytic at a point, then prove that its derivatives of all orders are analytic there.

b) Use Cauchy integral formula to evaluate $\int_C \frac{(z-3)\,dz}{(z^2+2z+5)}$ where C is the negatively oriented circle $|z+1-i| = 2$.

c) Prove that an entire bounded function is constant.

3+3+2

6. a) Derive the Taylor series representation of $\frac{1}{1-z}$ at $z_0 = i$.

b) State Laurent's theorem. Derive Taylor's theorem as a special case of Laurent's theorem.

3+(2+3)

7. a) Find the Laurent series expansions of $\frac{1}{z^3(1+z)}$ around the origin in two different regions.

b) Use Cauchy Residue Theorem to evaluate $\int_C \frac{dz}{z^3(z+4)}$ where C is the negatively oriented circle $|z + 2| = 3$. (2+2)+4

8. a) Using residues evaluate $\int cosec\, z\, dz$ over the positively oriented circle $|z| = 1.5$.

b) Find the nature of the singularity of the point $z_0 = 0$ with respect to the function $\frac{1-\cosh z}{z^3}$.

c) Use Rouche's theorem to determine the number of roots, counting multiplicities of the equation $z^4 - 2z^3 + 9z^2 + z - 1 = 0$ inside the region $1 \le |z| < 2$. 3+2+3

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